Analysis I   Lecture 8

Fundamental theorem of algebra

Let $a_0, a_1, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$ be coefficients of a polynomial

$$p(z) = \sum_{i=0}^{n} a_i \cdot z^i = a_0 + a_1 \cdot z + a_2 \cdot z^2 + \dots + a_n \cdot z^n$$

Then there exist $\underbrace{z_1, \dots, z_n}_{\text{roots of } p(z)} \in \mathbb{C}$ s.t.

$$p(z) = a_n \cdot \prod_{k=1}^{n} (z - z_k)$$

variable

fixed complex #s.

In particular, $p(z_j) = 0$ for any $j = 1 \dots n$

Indeed $p(z_1) = a_n \cdot \prod_{i=1}^{n} (z_1 - z_i)$

$= a_n \cdot \underbrace{(z_1 - z_1)}_{0} \times (z_1 - z_2) \times \dots \times (z_1 - z_n)$

$= 0.$

In other words if

$$p(z) = a_n \cdot \prod_{k=1}^{n} (z - z_k) \quad \text{then}$$

$z_1 \dots z_n$ are root (solutions)

of polynomial equation

$$\underline{p(z) = 0}.$$

Remark If $p(a) = 0$ then we can factorize $p(z)$ as

$$p(z) = (z-a) \cdot q(z)$$

for some polynomial $q(z)$.

Factorisation can be computed by long division.

Example $p(x) = x^4 - 5x^3 - 3x^2 + 17x - 10$

$$p(1) = 1 - 5 - 3 + 17 - 10 = 0$$

So $$p(x) = (x-1) \cdot q(x)$$

How to find $q(x)$?

$$\overbrace{x^4 - 5x^3 - 3x^2 + 17x - 10}^{p(x)} \;\Big|\; x - 1$$

$$\underbrace{x^3 - 4x^2 - 7x + 10}_{q(x)}$$

$$-\;(x^4 - x^3)$$

$$-4x^3 - 3x^2 + 17x - 10$$

$$-\;(-4x^3 + 4x^2)$$

$$-7x^2 + 17x - 10$$

$$-\;(-7x^2 + 7x)$$

$$10x - 10$$

$$-\;(10x - 10)$$

$$0$$

$$(x-1)\cdot x^3 = x^4 - x^3$$

$$(x-1)\cdot(-4x^2) = -4x^3 + 4x^2$$

$$(x-1)\cdot(-7x) = -7x^2 + 7x$$

$$p(x) = (x-1)\,q(x)$$

$$q(x) = x^3 - 4x^2 - 7x + 10$$

$$q(1) = 1 - 4 - 7 + 10 = 0 \Rightarrow$$

$$\Rightarrow q(x) = (x-1) \cdot \underline{h(x)}$$

can find h

by long division

$\deg(h) = 2 \Rightarrow$ roots of h are given by quadratic formula

$p(x) = x^4 - 5x^3 - 3x^2 + 17x - 10$

So we get that

$$p(x) = (x-1) \cdot q(x) =$$

$$= (x-1) \cdot (x-1) \cdot \underline{\underline{h(x)}} = \underline{\underline{(x-1)^2}} \cdot (x-5) \cdot (x-(-2))$$

$(x+2)$

$\shortparallel$

quadratic with roots 5 and -2

Fundamental thm of algebra
(alternative formulation)

For every polynomial $p(z)$ of degree > 0
the equation $p(z) = 0$ has a root
in $\mathbb{C}$.

i.e. there exists complex number
a s.t. $p(a) = 0$.

If $p(a) = 0 \Rightarrow p(z) = (z-a)\,q(z)$

by Remark

So we can find a new root of $p(z)$

by finding a root of $q(z)$

Since $q(b) = 0 \Rightarrow p(b) = (b-a)\cdot \underset{0}{\underset{\shortparallel}{q(b)}} = 0$

Roots of polynomials with real coeff.

$$ax^2 + bx + c = 0 \quad \text{with } a, b, c \in \mathbb{R}$$

$$\Rightarrow \quad x_1 = \frac{-b + \sqrt{\underbrace{b^2 - 4ac}_{\Delta}}}{2a} \; ; \; x_2 = \frac{-b - \sqrt{\underbrace{b^2 - 4ac}_{\Delta}}}{2a}$$

if $\Delta \geq 0$ we have 2 real roots

if $\Delta < 0$ then $x_1 = \overline{x_2}$

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<u>Example</u> $x^2 + x + 1 = 0$

$\Delta = b^2 - 4ac = 1 - 4 \cdot 1 \cdot 1 = -3 < 0$

$\sqrt{\Delta} = \sqrt{3} \cdot i$ or $-\sqrt{3}\, i$

So, we get $x_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} \cdot i$ $x_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} \cdot i$

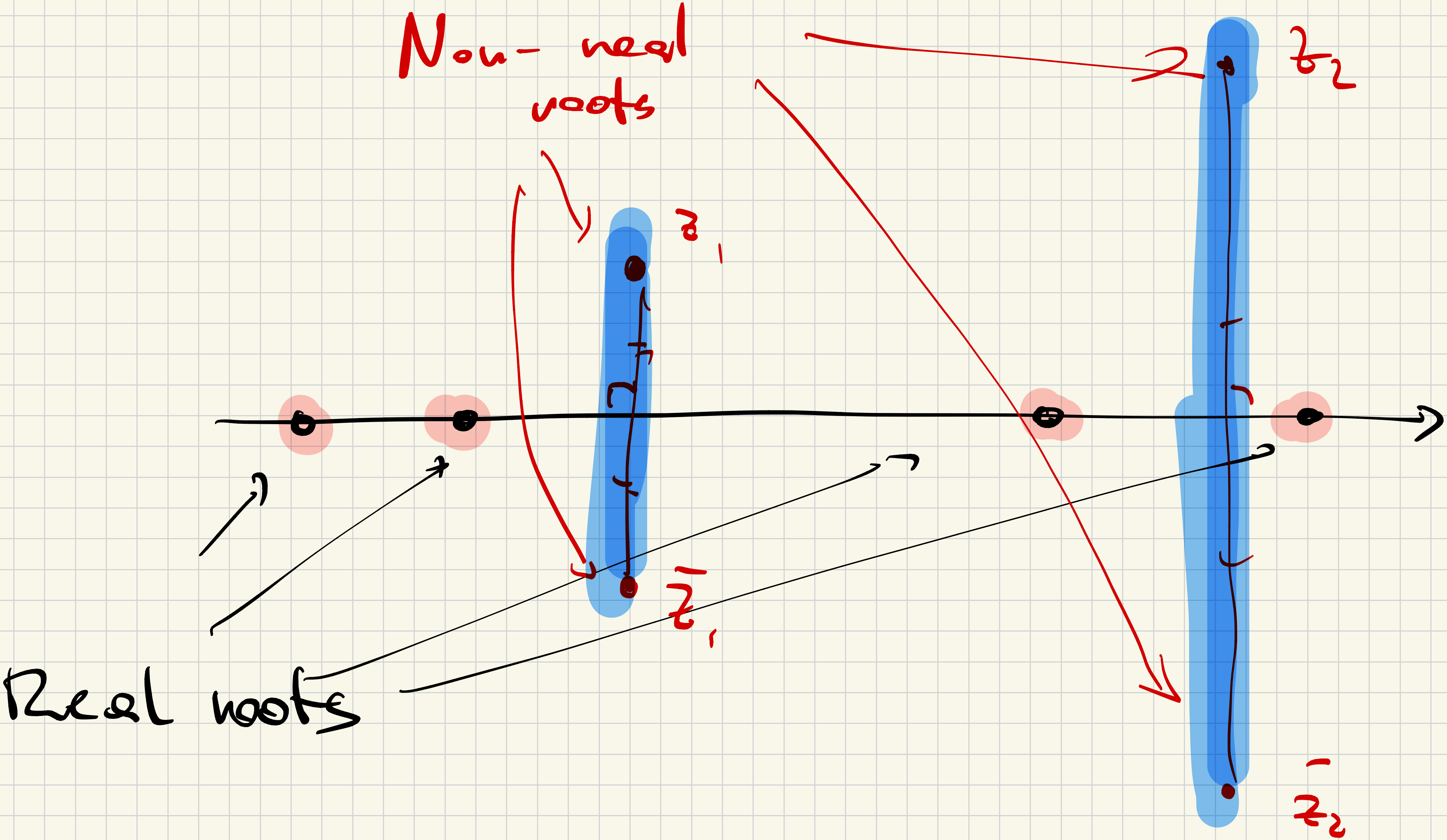
Fundamental thm of algebra over $\mathbb{R}$

Let $p(x)$ be a polynomial with coefficients in $\mathbb{R}$ then

easy to show by Analysis
1) If degree of $p(x)$ is odd then $p(x)$ has a real root

2) If a complex number z is a root then $\bar{z}$ is also a root.

Roots of real polynomials:



Alternative form:

Any real polynomial $p(x)$ factors

as

$$p(x) = \prod (x - x_i) \cdot \prod \underbrace{q_j(x)}_{\substack{\text{quadratic} \\ \text{polynomials} \\ \text{(degree 2)}}}$$

x_i - real root of $p(x)$

Sequences of real numbers

A sequence is an infinite string of numbers:

$$x_0,\ x_1,\ x_2,\ x_3 \ldots \qquad x_i \in \mathbb{R}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \\ 0 & 1 & 2 & \ldots \end{matrix}$$

More formally a sequence is a map (or function) $f: \mathbb{N} \longrightarrow \mathbb{R}$

$$n \longmapsto x_n$$

We will denote the whole sequence

by $(x_n)_{n \geq 0}$.

Sometimes we would like to index elements of sequence with some $n_0 \geq 1$

$$(x_n)_{n \geq n_0} = x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots$$

Examples

1) Harmonic sequence $(x_n)_{n\geq 1}$

defined by $x_n = \frac{1}{n}$

$x_1 = 1, \quad x_2 = \frac{1}{2}, \quad x_3 \frac{1}{3}, \ldots$

(Alternatively consider $(x_n)_{n\geq 0}$

with $x_n = \frac{1}{n+1}$)

2) Alternating harmonic sequence

$(x_n)_{n \geq 1}$ defined by $x_n = (-1)^{n+1} \frac{1}{n}$

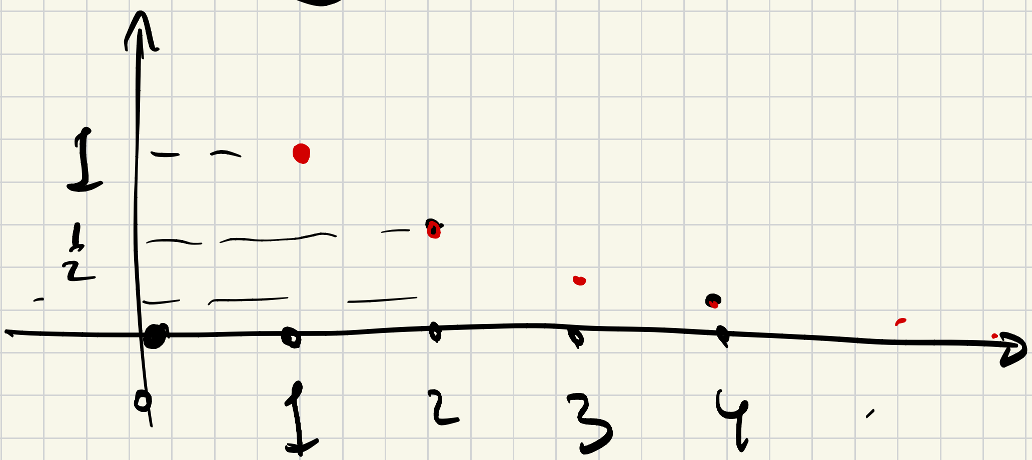
$$x_1 = (-1)^{1+1} \cdot \frac{1}{1} = 1$$

$$x_2 = (-1)^{2+1} \cdot \frac{1}{2} = -\frac{1}{2}$$

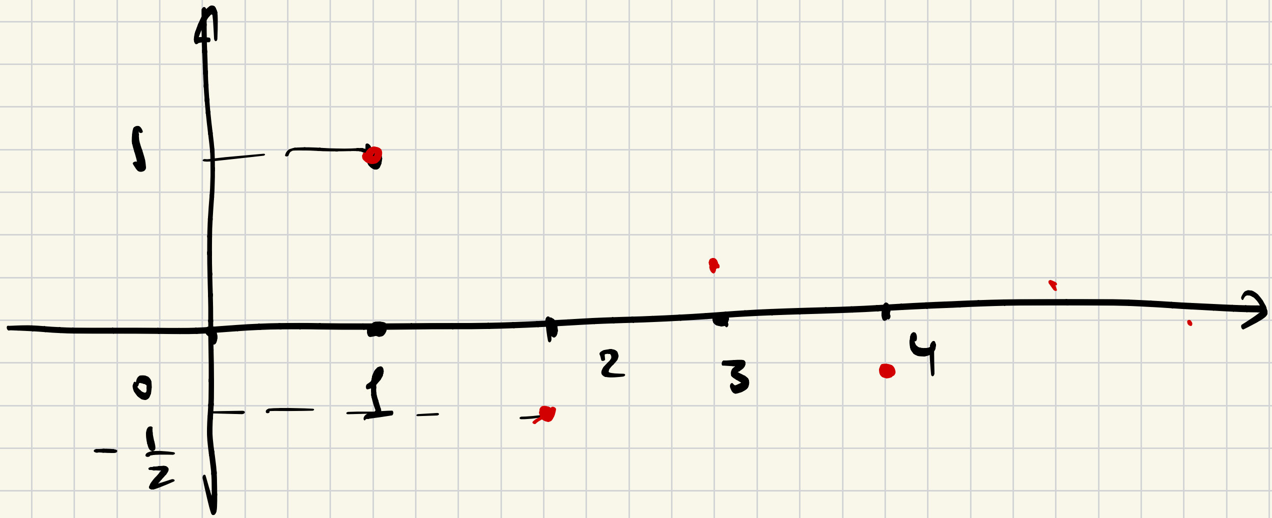
$x_3 = \frac{1}{3}$ $\qquad$ $x_4 = -\frac{1}{4}$

Plot of harmonic and alternating harmonic sequences

harmonic



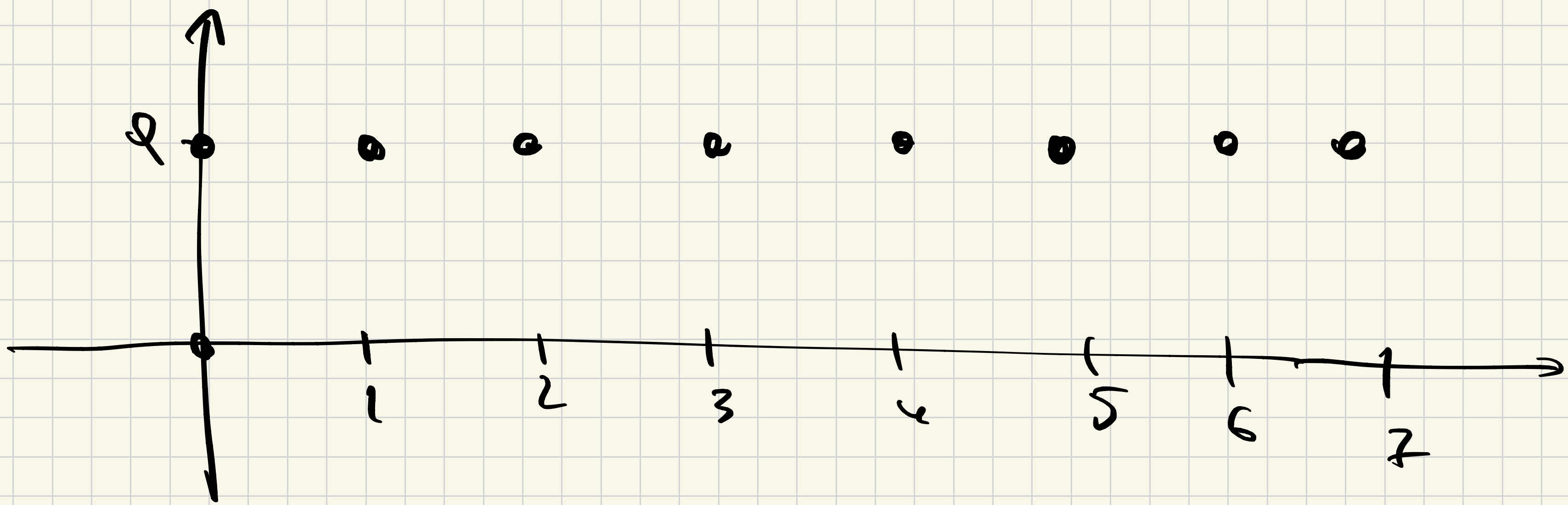
Alternating harmonic



3) Constant sequence

Fix some $a \in \mathbb{R}$

$X_n = a$ for $n \geq 0$



Another way to define (x_n) is by recursive relation:

Examples: Arithmetic sequence:

$x_0 = a$ and $x_n = x_{n-1} + b$ for some $a, b \in \mathbb{R}$

this can also be defined by $x_n = a + n \cdot b$.

Example **Geometric sequence**

Fix $a, b \in \mathbb{R}$ then

$x_0 = a$ and $x_n = x_{n-1} \cdot b \quad n \geq 1$

recursive definition

OR defining by a formula

$$x_n = a \cdot b^n \qquad x_0 = a \cdot b^0 = a$$

Anothe recursive sequence is

Fibonacchi sequence:

$x_0 = 1, \quad x_1 = 1$ and

for $n \geq 2$ $\quad x_n = x_{n-1} + x_{n-2}$

$x_0 = 1$; $x_1 = 1$; $x_2 = x_1 + x_0 = 2$; $x_3 = 2+1 = 3$; $x_4 = 5, \ldots$

Question: Is there explicit formula for x_n?

4) <u>Sequence of primes</u> $(x_n)_{n\geq 1}$

x_n is the n-th positive prime number

E.g.

$x_1 = 2 \qquad x_5 = 11$

$x_2 = 3 \qquad x_6 = 13$

$x_3 = 5 \qquad \vdots$

$x_4 = 7$

Definition let $(x_n)_{n \geq n_0}$ be a sequence

then we say that

1) (x_n) is increasing if $x_{n+1} \geq x_n \quad \forall n \geq n_0$

2) (x_n) is strictly increasing if $x_{n+1} > x_n \quad \forall n \geq n_0$

3) (x_n) is decreasing if $x_{n+1} \leq x_n \quad \forall n \geq n_0$

4) (x_n) is strictly decreasing if $x_{n+1} < x_n \quad \forall n \geq n_0$

5) (x_n) is (strictly) MONOTONE

if it is (stricly) decresing or

(stricly) incresing.

6) (x_n) is bounded above if $\exists M \in \mathbb{R}$

s.t. $x_n < M \quad \forall n \geq n_0$

7) (x_n) is bounded below if $\exists m \in \mathbb{R}$

s.t. $x_n > m \quad \forall n \geq n_0$

Remark

$(x_n)_{n \geq n_0}$ is bounded (below or above)

if $\{x_n \mid n \geq n_0\}$ is bounded

(below or above).

Notation

$\sup(x_n)_{n \geq n_0}$ and $\inf(x_n)_{n \geq n_0}$

By $\inf(x_n)$ we really mean

$$\inf \{ x_n \mid n \geq n_0 \}$$

By $\sup(x_n)$ we mean

$$\sup \{ x_n \mid n \geq n_0 \}.$$

Proposition (x_n) is bounded if and only if

$$\exists\, c \geq 0 \text{ s.t. } \forall n \quad |x_n| < c.$$

i.e. bounded below and above.

Proof: " $\Rightarrow$ "

(x_n) is bounded

Need to show $\Rightarrow \exists\, c \in \mathbb{R}$ s.t. $|x_n| \leq c$:

(x_n) is bounded $\Rightarrow \exists\, m, M$ s.t. $m < x_n < M \ \forall n$

$\Rightarrow$ we can take $c = \max(|m|, |M|)$.

then we will have $|x_{n_0}| \leq C = \max(|M|, |m|)$

" $\Leftarrow$ "

Need to show

If $\exists$ $C \geq 0$ s.t. $|x_n| \leq C$ $\forall n$ $\Rightarrow$

$\Rightarrow$ (x_n) is bounded.

If C is such that $|x_n| \leq C$ $\forall n$

then $M = C$ is an upper bound of (x_n)

and $m = -C$ is a lower bound of (x_n)

So (x_n) is bounded

∎

Example Prove that (x_n) defined by

$$x_n = (-1)^n \frac{1}{1+\sqrt{n}} \quad \text{is bounded.}$$

Proof It is enough to check that

$$\exists \; c > 0 \quad \text{s.t.} \quad |x_n| \leq c$$

But $|x_n| = \dfrac{1}{1+\sqrt{n}}$ since $1+\sqrt{n} \geq 1$

then $\dfrac{1}{1+\sqrt{n}} \leq 1 \Leftrightarrow c = 1$ works $\blacksquare$